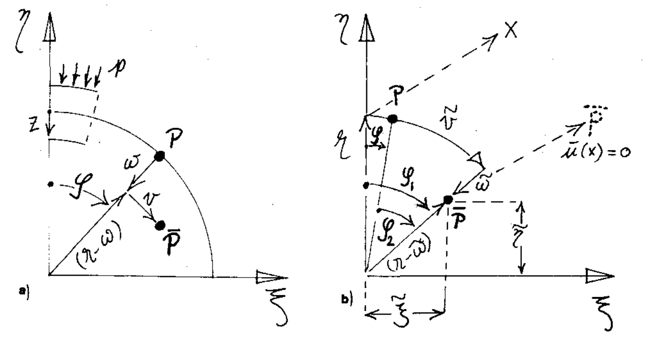

Fig. 1 Coordinate system $(r,\varphi,S=x/r)$ and displacement field W of the circular cylindrical shell; a) in the case of tangential (vectorial) v definition $W(\omega,v,\bar{u})$, b) in the case of circumferential $\tilde{v}$ definition $\tilde{W}$ $(\tilde{\omega},\tilde{v},\bar{u})$.

which is a function of only one unknown $(\tilde{\omega})$. On linearizing Eq. (32), the exact linear part is

$$\tilde{\chi}_L = (\ddot{\tilde{\omega}}+\tilde{\omega})/r^2 \tag{33}$$

which is obviously far simpler than the correspondant in the case of tangential v definition of Eq. (31). Similarly simple and accurate expressions, although approximate, can be obtained in the case of a tangential v definition only via Koiter-John's criteria of error estimation.[8-12,13]

Appendix 1 The Derivation of Eq. (21)

Expanding Eq. (17) we get

$$\begin{aligned}\tilde{\varepsilon}_\varphi = 1+\frac{1}{2}\Big[&-\frac{2\tilde{\omega}}{r}+\left(\frac{\tilde{\omega}}{r}\right)^2+\left(\frac{\tilde{v}}{r}\right)^2-4\frac{\tilde{v}^\bullet\bar{\omega}}{r}+\left(\frac{\dot{\tilde{\omega}}}{r}\right)^2\\ &+\text{third and fourth-order terms}\\ &-\frac{1}{8}\Big[-\frac{2\tilde{\omega}}{r}+\frac{2\tilde{v}}{r}+\text{second, third, and fourth-order terms}\\ &+\text{terms of higher order}\Big]^2+\ldots\end{aligned} \tag{A1}$$

Since we have neglected third and fourth-order terms in the first bracket of Eq. (A1) and enforced a restriction on the magnitude of the displacement by assuming that every term must be sufficiently small so that the terms' sum remains smaller than unity (Eq. 20), we can consequently expand both brackets of Eq. (A1). Thus, we obtain

$$\bar{\varepsilon}_\varphi = \frac{1}{r}(\tilde{v}-\tilde{\omega})+\frac{1}{2}\left\{\frac{(\dot{\tilde{\omega}})^2}{r}-2\frac{\tilde{\omega}\tilde{v}}{r^2}\right\} \tag{A2}$$

which confirms Eq. (21).

Appendix 2 The Derivation of Eq. (17)

Following Fig. 1b we have

$$\varphi_1 = \varphi+\varphi_2 = \varphi_+\tilde{v}/r$$

Thus

$$\mathrm{d}\varphi_1 = \mathrm{d}\varphi\left(1+\frac{\tilde{v}^\cdot}{r}\right);\ \ (\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}\varphi \tag{A3}$$

We see further that

$$\bar{\xi} = (r-\omega)\sin\varphi_1,\ \ \tilde{\eta} = (r-\omega)\cos\varphi_1$$

Thus

$$(\bar{\xi}')^2+(\tilde{\eta}')^2 = \tilde{\omega}'^2+(r-\omega)^2;\ \ (\ ') = \mathrm{d}(\ )/\mathrm{d}\varphi_1 \tag{A4}$$

Since the length of an element $\mathrm{d}s = r\mathrm{d}\varphi$ after deformation is

$$\overline{\mathrm{d}S^2} = (\mathrm{d}\bar{\xi})^2+(\mathrm{d}\tilde{\eta})^2$$

we obtain

$$(\overline{\mathrm{d}S}/\mathrm{d}\varphi_1)^2 = (\tilde{\xi}')^2+(\tilde{\eta}')^2 = \tilde{\omega}'^2+(r-\tilde{\omega})^2 \tag{A5}$$

Considering Eqs. (A3) and (A4), the strain becomes

$$\begin{aligned}\tilde{\varepsilon}_{\varphi_1} &= \frac{\overline{\mathrm{d}S}}{dS}-1 = \left[\tilde{\omega}'^2+(r-\tilde{\omega})^2\right]^{1/2}\frac{d\varphi_1}{rd\varphi}-1\\ &= \frac{[\tilde{\omega}'^2+(r-\tilde{\omega})^2]^{1/2}}{r-v'}-1\end{aligned} \tag{A6}$$

With the aid of (A3) and the transformations

$$\tilde{\omega}' = \mathrm{d}\tilde{\omega}/\mathrm{d}\varphi_1 = \frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}\varphi(1+\tilde{v}^\bullet/r)} = \frac{\tilde{\omega}^\bullet}{(1+\tilde{v}^\bullet/r)}$$

and $\tilde{v}' = \tilde{v}^\bullet/(1+\tilde{v}^\bullet/r)$ the strain can be written in terms of the original co-ordinates as

$$\tilde{\varepsilon}_\varphi = \left[\left(\frac{\tilde{\omega}^\bullet}{r}\right)^2+\left\{\left(1-\frac{\tilde{\omega}}{r}\right)\left(1+\frac{\tilde{v}^\bullet}{r}\right)\right\}^2\right]^{1/2}-1$$

which confirms Eq. (17).

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Theoretical Analysis for the Burning of Solid Propellants

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Nomenclature

c = specific heat of solid

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c_p = specific heat of gas at constant pressure
E = activation energy
F = film coefficient
k_0 = reaction constant
L = latent heat of evaporation
m = mass flux rate
n = pressure exponent
p = pressure
Q = heat released per unit mass of reacting material
r = linear burning rate
R = gas constant
T = absolute temperature
t = time
W = volumetric reaction rate
x = space coordinate
y = dimensionless space coordinate
λ = thermal conductivity
ρ = density
ϕ_0 = dimensionless simplex

Subscripts

c = condensed phase
f = gas flame
s = surface

Introduction

COMPOSITE propellants are now widely used in solid fuel rocket chamber. However, the burning rate prediction of these propellants is still an unsolved problem. The problem is obviously complicated because many hetrogeneous physicochemical processes are involved during combustion. The burning of the solid fuels is characterized by the diversity of phenomena which occur during combustion.

Geckler,[1] Green,[2,3] Rice and Ginnel,[4] and Par and Crawford[5] studied the phenomenon of combustion in the solid propellants but these works cover only a part of the whole process. Sotter[6] considered the possible reaction mechanism and associated kinetics; however, his study was limited to the gaseous phase only. Besides, he did not consider the transport mechanism in the gaseous phase and he calculated the burning rate for pressure only. Culick[7] and Culick and Dehority[8] performed a detailed analysis for the combustion of solid propellants.

This Note is concerned with a theoretical investigation of the process of combustion in a rocket chamber. The temperature profiles for the downstream, upstream and reaction zone are determined. The analysis is further extended to deduce the stability limit for the propellant combustion.

Formulation

Consider a simplified model of an unidimensional and nonadiabatic flame. For simplicity, we shall ignore the influences of pressure and viscous forces and we shall assume the mass diffusivity of any constituent equal to the thermal diffusivity of the mixture (i.e., Lewis number is unity). We shall also neglect the turbulence effect. Thus the transient temperature distribution is determined by the energy equation

$$\rho c_p\left(\frac{\partial T}{\partial t}+r\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+WQ \tag{1}$$

complemented with the boundary conditions as follows: At a sufficient distance from the propellant surface, the chemical reaction is completed at flame temperature T_f corresponding to total conversion of chemical energy into thermal energy

$$T=T_f \tag{2a}$$

$$\frac{\partial T}{\partial x}\rightarrow 0;\quad x\rightarrow\infty \tag{2b}$$

and at the surface of the solid propellant the temperature is equal to T_s, the time rate flow of heat into the propellant corresponds to the heat expanded in gasifying the propellant[3,10]

$$T=T_s \tag{3a}$$

$$\frac{F}{r}(T_f-T_s)=\rho\gamma L+\left(\lambda\frac{\partial T}{\partial x}\right)\Big|_s \tag{3b}$$

In the reaction zone, the temperature profile and its gradient must be continuous. For a zero-order chemical reaction

$$W=k_0p^n\exp(-E/RT) \tag{4}$$

and following to Zel'dovich-Frank-Kamenetsku theory[9] the burning rate is given by the expression

$$m^2=\frac{2Q\lambda}{c^2(T_s-T_c)^2}\int_{T_c}^{T_s}WdT \tag{5}$$

Equation (1) is a nonlinear partial differential equation which is here tackled by the method of perturbation technique. The nonsteady variation in temperature and burning rate are expressed by small perturbations[3,10]

$$T=T_0(x)+\Delta T(x,t) \tag{6a}$$

$$T_s=T_{s0}+\Delta T_S(t) \tag{6b}$$

$$r=r_0(T_{s0})+\Delta r(\Delta T_s) \tag{6c}$$

$$W=W_0(T_0)+\Delta W(\Delta T) \tag{6d}$$

On substituting these equations into Eq. (1) and collecting zero-order term, we have

$$\rho r_0 c_p\frac{dT_0}{dx}=\frac{d}{dx}\left(\lambda\frac{dT_0}{dx}\right)+W_0Q \tag{7}$$

which describes the steady-state heat conduction problem. The boundary conditions are

$$T_0=T_f \tag{8a}$$

$$\frac{dT_0}{dx}\rightarrow 0,\quad x\rightarrow\infty \tag{8b}$$

and

$$T_0=T_{s0} \tag{9a}$$

$$\frac{F_0}{r_0}(T_f-T_{s0})=\rho r_0L-\left(\lambda\frac{dT_0}{dx}\right)_s \tag{9b}$$

at the surface.

On introducing a variable y

$$y=\exp\left(\int_0^x\frac{\rho r_0c_p}{\lambda}\mathrm{d}x\right);\quad\frac{dy}{dx}=\frac{\rho r_0c_p}{\lambda}y$$

Eq. (7) is changed as

$$y^2\frac{d^2T_0}{dy^2}=-\tfrac{1}{2}(c/c_p)^2(T_{s0}-T_c)^2\phi_0 \tag{10}$$

where we have introduced a dimensionless simplex

$$\phi_0=\frac{W_0}{\bar{W}};\quad\bar{W}=\frac{1}{T_s-T_c}\int_{T_c}^{T_s}Wdt$$

The boundary conditions follow accordingly.

For the downstream and upstream region, the function vanishes and we are left with

$$y^2\frac{d^2T_0}{dy^2}=0 \tag{11}$$

Equation (11) is integrated and after using the corresponding boundary conditions, we find the temperature profile

$$T_0(y)=T_f \tag{12}$$

for the downstream region and

$$T_0(y)=T_{s0}-(1-y)\left[\frac{L}{c_p}+\frac{F_0}{c_p\rho r_0{}^2}(T_f-T_{s0})\right] \tag{13}$$

for the upstream region. The result of uniform temperature distribution in downstream region was to be expected. The temperature distribution and its gradient must be continuous in the reaction zone where Eq. (10) takes the form

$$y^2\frac{d^2T_0}{dy^2}=-\tfrac{1}{2}A\exp(-E/RT_0) \tag{14}$$

where

$$A=(c/c_p)^2\frac{(T_{s0}-T_c)}{\bar{W}}k_0p^n$$

Equation (14) is integrated and using the mean value of integration, we can write,

$$\left(\frac{dT_0}{dy}\right)^2=\frac{A}{y}\int_{T_{s0}}^{T_0}\exp(-E/RU)du \tag{15}$$

In view of the continuity of temperature and its gradient, Eq. (15) is further integrated as

$$\int_{T_{s0}}^{T_0}\frac{du}{\sqrt{[u\exp(-E/RU)+\frac{E}{R}Ei(-E/RU)-T_{s0}\exp(-E/RT_{s0})-\frac{E}{R}Ei(-E/RT_{s0})]}}=2(c/c_p)\frac{T_{s0}-T_c}{\bar{W}}k_0^{1/2}p^{n/2}(\sqrt{y}-1) \tag{16}$$

where

$$Ei((x)=\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$$

is an integral exponential function.

The solution of Eq. (16) gives the temperature distribution in the reaction zone. Temperature profile given in Eq. (16) is analyzed for different parameters. Computed values of the flame thickness versus pressure are plotted in Fig. 1 for $\eta=0.413$ and a comparison is made with the result of Culick and Dehority.[8] For low pressure the flame thickness increases more in the present analysis while it is comparable with the result of Culick[7] and Dehority[8] in the higher pressure range. Further, by the increase of reaction constant, the flame thickness decreases, i.e., the reaction zone becomes thinner. The thinner reaction zone means higher ignition temperature and thus the burning rate is enhanced.

The nonsteady part of the temperature distribution is described by the first-order perturbation term. In order to study the stability we consider a periodic variation[10] in the film coefficient factor and temperature, i.e.

$$\Delta F=F_1e^{st},\ \Delta T=f(y)e^{st},\ s=\sigma+i\omega \tag{17}$$

and, if the variation in burning rate follows the variation in surface temperature after a time lag τ, required for the completion of the phase change then

$$\Delta r=\frac{dr}{dT_s}\bigg|_{T_{s0}}\Delta T_s(t-T)=\chi f(1)\exp^{[s(t-\tau)]} \tag{18}$$

where

$$\chi=\sqrt{\frac{2Q\lambda}{c^2}}\left[-\frac{1}{(T_{s0}-T_c)^2}\left(\int_{T_c}^{T_{s0}}WdT\right)^{1/2}\right.$$
$$\left.+\tfrac{1}{2}\frac{W(T_{s0})}{(T_{s0}-T_c)}\left(\int_{T_c}^{T_{s0}}WdT\right)^{-1/2}\right]$$

Proceeding on the line of Green[3] and after some manipulation, the first-order perturbation of Eq. (1) becomes

$$y^2\frac{d^2f}{dy^2}-\frac{s\lambda}{c_p\rho r_0{}^2}f=\chi f(1)\left[\frac{L}{c_p}\right.$$
$$\left.-\frac{F_0}{c_p\rho r_0{}^2}(T_f-T_{s0})\right]\frac{1}{\rho r_0{}^2}y\exp^{(-s\tau)} \tag{19}$$

and

$$\frac{df(1)}{dy}-\left[\left\{\frac{L}{c_p}+\frac{F_0}{c_p\rho r_0{}^2}(T_f-T_{s0})\right\}\frac{\chi}{r_0}\exp^{(-s\tau)}\right.$$
$$\left.+\frac{F_0}{c_pr_0{}^2}\right]f(1)+\frac{F_1}{c_pr_0{}^2}(T_f-T_{s0})=0 \tag{20}$$

In order to solve Eq. (19) an extra condition must be imposed that, in the solid material, all perturbations must be damped.

Equation (19) is integrated as

$$f(y)=A_1y^{\alpha+}+A_2y^{\alpha-}$$
$$-\left[L-\frac{F_0}{\rho r_0{}^2}(T_f-T_{s0})\right]\frac{\chi f(1)}{s\lambda}y\exp^{(-s\tau)} \tag{21}$$

where A_1 and A_2 are constants of integration and

$$\alpha_\pm=\tfrac{1}{2}\left[1\pm\sqrt{\left(1+4\frac{s\lambda}{c_p\rho r_0{}^2}\right)}\right] \tag{22}$$

Here the coefficient A_1 vanishes due to damping condition and the coefficient A_2 is determined by the boundary condition (20). On substituting the value of A_2 so obtained in Eq. (28), the amplitude of the surface temperature variation is obtained as

$$f(1)=\frac{F_1}{c_pr_0{}^2}(T_f-T_{s0})[\phi_1+i\phi_2] \tag{23}$$

where

$$\phi_i=\frac{\psi_i}{\psi_1{}^2+\psi_2{}^2}$$

$$\psi_1=1-\frac{F_0}{c_pr_0{}^2}+\left[L+\frac{F_0}{\rho r_0{}^2}(T_f-T_{s0})\right]\frac{\chi}{c_pr_0}\cos\omega\tau\exp(-\sigma\tau)$$
$$-\frac{M}{(\sigma^2+\omega^2)}\left[L-\frac{F_0}{\rho r_0{}^2}(T_f-T_{s0})\right]$$

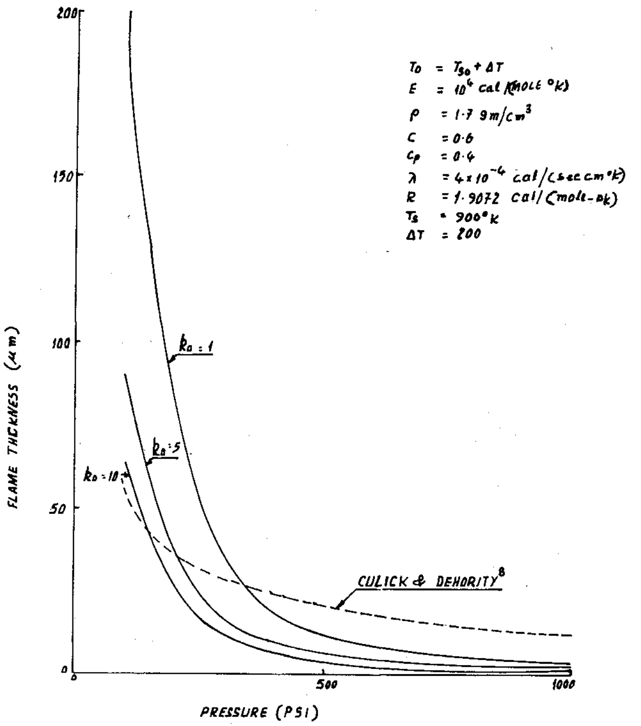

Fig. 1 **Dependence of flame thickness on pressure and reaction constant.**

$$\times \frac{\chi}{\lambda}(\omega\cos\omega\tau + \sigma\sin\omega\tau)\exp(-\sigma\tau)$$

$$\psi_2 = -\left[L + \frac{F_0}{\rho r_0{}^2}(T_f - T_{s0})\right]\frac{\chi}{c_p r_0}\sin\,\omega\tau\,\exp(-\sigma\tau)$$

$$+\frac{M}{(\sigma^2+\omega^2)}\exp(-\sigma\tau)\left[L - \frac{F_0}{\rho\gamma_0{}^2}(T_f - T_{s0})\right][\sigma\cos\omega\tau$$

$$-\omega\sin\omega\tau]\,\theta = ½\tan^{-1}\left[4\frac{\omega\lambda}{\rho c_p r_0{}^2 + 4\sigma\lambda}\right], \beta = \left[\left(1 + 4\frac{\sigma\lambda}{\rho c_p r_0{}^2}\right)^2\right.$$

$$\left.+\frac{16\omega^2\lambda^2}{\rho^2 c_p^2 r_0{}^4}\right]^{¼} \quad M = \frac{2\beta\sin\theta}{1+\beta^2-2\beta\cos\sigma}$$

Equation (23) gives the amplitude of the surface temperature variation and thus the small perturbation in surface temperature is given by

$$\Delta T_s(t) = \frac{F_1}{c_p r_0{}^2}(T_f - T_{s0})(\psi_1{}^2 + \psi_2{}^2)^{-½}$$

$$\times\exp\left[i\left(\omega t + \tan^{-1}\frac{\psi_2}{\psi_1}\right)\right] \tag{24}$$

The positive real parts of the solution (24) indicate unstable behavior. To investigate stability limit, the real part of the solution is equated to zero, which gives

$$\psi_2 + \psi_1 \cot\,\omega t = 0 \tag{25}$$

Equation (25) gives the necessary condition for stability. By varying the parameters, the influences of changes in physical parameters of the composition on unstable burning can be predicted.

Conclusion

Perturbation technique has been employed to investigate the transient process of combustion of solid propellant. We have presented a discussion on quantities which are not measurable, for example, the flame thickness and chemical reaction rate. Variation of these quantities are shown graphically. The advantage of this analysis is that it enables one to determine quite easily how the various parameters influence the combustion mechanism.

Secondly, a range for stability in terms of known parameters is deduced which can help designers to predict the influence of changes in propellant's parameters on the unstable burning.

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Technical Comments

Comment on "Optimization of Ring Stiffened Cylindrical Shells"

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RELIABILITY is a major problem associated with nonlinear mathematical programming (NLP) optimization methods. Eason and Fenton[1] in a recent comparison study of some 17 NLP codes found that a typical NLP code solved only half of the ten test problems studies. Thus the user of such methods cannot assume that merely because he applies some popular NLP procedure to a given design problem it will generate optimal solutions. Confirmation studies and/or some check on the optimality of the solutions are needed to establish that the NLP method selected indeed works on the problem to which it has been applied. Any

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